Self-consistency in relativistic theory of infinite statistics fields

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Infinite statistics in which all representations of the symmetric group can occur is known as a special case of quon theory. Our previous work has built a relativistic quantum field theory which allows interactions involving infinite statistics particles. In this paper, a more detailed analysis of this theory is available. Topics discussed include cluster decomposition, CPT symmetry and renormalization.

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I. INTRODUCTION

Most conventional quantum theories are based on Bose-Einstein statistics and/or Fermi-Dirac statistics, i.e. only two 1-dimensional representations of the permutation group are allowed. However the general principles of quantum theory do not have this requirement [1], and Bose (Fermi) statistics can be violated by a small amount. One famous approach of such violation is quon theory, in which the basic algebra can be obtained as the convex sum of Bose and Fermi statistics [2]. Especially when the sum is the average of these two algebras, we can get a special statistics called infinite statistics [3].

The basic algebra of infinite statistics is $a_k a_l^\dagger = \delta_{kl}$, which involves no commutation relation between annihilation and creation operators. The quantum states are orthogonal under any permutation of the identical particles. So it allows all representations of the symmetric group to occur. Furthermore, the loss of local commutativity also implies violation of locality, which is an important characteristic of quantum gravity. By virtue of these properties, infinite statistics has been applied to many subjects, such as black hole statistics [4–6], dark energy quanta [7–11], large N matrix theory [12–14] and holography principle [15, 16]. Many of these applications involve discussions in relativistic case.

However there are some difficulties for infinite statistics to have a consistent relativistic theory. Due to the lack of local commutativity, Lorentz invariance of the S-matrix is unapparent in the Dyson formula [2, 3]. And the conservation of statistics also acquires some special form the the interaction Hamiltonian [17]. Our previous work [18] constructs a relativistic quantum field theory obeying infinite statistics by solving the two difficulties above.

In the present work we keep on investigating the self-consistency of this theory. Greenberg has showed that cluster decomposition and the CPT theorem hold for free fields [2]. In this paper, we show that the clustering still holds for interacting theory, but CPT symmetry violates in vector-spinor interactions. We also discuss the renormalizability of this theory by using non-perturbative methods.

This paper is organized as follows. In Sec. II we introduce the elementary ingredients of infinite statistics and main results of our previous work. In Sec. III we prove the cluster decomposition for our new theory. In Sec. IV we discuss the CPT symmetry. In Sec. V we discuss the renormalizability of infinite statistics theory. General conclusions are given in Sec. VI. We also introduce a non-perturbative method for infinite statistics field theory in Appendix.

II. INFINITE STATISTICS

The quon algebra is the convex sum of Bose and Fermi statistics

$$\frac{1+q}{2}[a_k, a_l^\dagger] + \frac{1-q}{2}[a_k, a_l^\dagger]_+ = \delta_{kl}, \tag{1}$$

convexity requires $0 \leq q \leq 1$. When $q = +1$ ($q = -1$), this becomes Bose (Fermi) statistics. The quon statistics interpolates smoothly between Bose and Fermi statistics when q transfers from $+1$ to -1. Especially when $q = 0$ each statistics has equal weight and all representations of the symmetric group can occur. We call it infinite statistics, in which the basic algebra is

$$a_k a_l^\dagger = \delta_{kl}. \tag{2}$$

We can get a Fock-state representation by defining a unique vacuum state annihilated by all the annihilators

$$a_k|0\rangle = 0. \tag{3}$$

The m-particle state is constructed as

$$|\phi_m\rangle = (a_{k_1}^\dagger)^{m_1}(a_{k_2}^\dagger)^{m_2}...(a_{k_j}^\dagger)^{m_j}|0\rangle \tag{4}$$

with $m_1+m_2+...+m_j = m$, where we have $a_{k_i}^\dagger \neq a_{k_{i+1}}^\dagger$. Such states have positive norms and the normalization factor equals one. Since there is no commutation relation between two annihilation or creation operators, the states created by any permutations of creation operators

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are orthogonal. That's why it is also called quantum Boltzmann statistics.

One can define a set of number operators $\hat{n}_i$ such that

$$[\hat{n}_i, a_j] = -\delta_{ij} a_j. \tag{5}$$

Then the total number operator is $N = \sum_i \hat{n}_i$, and the energy operator is given by $E = \sum_i \epsilon_i \hat{n}_i$, where ϵ_i is the single particle energy. The explicit form of $\hat{n}_i$ is

$$\hat{n}_i = a_i^\dagger a_i + \sum_k a_k^\dagger a_i^\dagger a_i a_k + \sum_{k_1,k_2} a_{k_1}^\dagger a_{k_2}^\dagger a_i^\dagger a_i a_{k_2} a_{k_1} + \cdots + \sum_{k_1,k_2,\ldots,k_s} a_{k_1}^\dagger a_{k_2}^\dagger \cdots a_{k_s}^\dagger a_i^\dagger a_i a_{k_s} \cdots a_{k_2} a_{k_1} + \cdots, \tag{6}$$

which is obviously a non-local operator. One can directly use relation (2) to check that this definition obeys Eq. (5).

All the above discussion is under non-relativistic case. It's not difficult to extend these to the relativistic field theory. We can construct infinite statistics field that transform irreducibly under the Lorentz group in the same way of conventional Bose (Fermi) fields construction. By using the general transformation rules of particle states and the Fourier transform, the annihilation field $\psi_l^+(x)$ and creation field $\psi_l^-(x)$ with mass m in momentum space are

$$\psi_l^{+(n)}(x) = \sum_{\sigma n} (2\pi)^{-3/2} \int d^3p \; u_l^{(n)}(\boldsymbol{p}, \sigma) e^{ip\cdot x} a_{\boldsymbol{p}}^{(n)}(\sigma), \tag{7}$$

$$\psi_l^{-(n)}(x) = \sum_{\sigma n} (2\pi)^{-3/2} \int d^3p \; v_l^{(n)}(\boldsymbol{p}, \sigma) e^{-ip\cdot x} a_{\boldsymbol{p}}^{\dagger(n)}(\sigma), \tag{8}$$

Here we have chosen relativistic four-vector notation $p^\mu = (p^0, \boldsymbol{p})$ $(p^0 \equiv \sqrt{\boldsymbol{p}^2 + m^2})$, σ labels spin z-components (or helicity for massless particles), and the superscript (n) labels particle species

$$a_{\boldsymbol{p}}^{(n)}(\sigma) a_{\boldsymbol{p}'}^{\dagger(n')}(\sigma') = \delta(nn')\delta(\sigma\sigma')\delta^3(\boldsymbol{p} - \boldsymbol{p}'). \tag{9}$$

In a theory based on infinite statistics the local commutativity ($[\psi(x), \psi^\dagger(y)]_\mp = 0$ for $x - y$ spacelike) does not hold, so we can't uniquely determine a linear combination field $\psi(x) = \kappa\psi^+(x) + \lambda\psi^{-c}(x)$, so the basic field in this theory should be $\psi^+(x)$ and $\psi^-(x)$.

For a relativistic field theory, the S-matrix should be Lorentz invariant,

$$\begin{aligned} S &= T\{\exp(-i \int_{-\infty}^{\infty} dt V(t))\} \\ &= 1 + \sum_{N=1}^{\infty} \frac{(-i)^N}{N!} \int d^4x_1 \cdots d^4x_N T\{\mathscr{H}(x_1) \cdots \mathscr{H}(x_N)\}, \end{aligned} \tag{10}$$

In which $V(t) = \int d^3x \mathscr{H}(x)$ is the interaction term $H = H_0 + V$ and $T\{\;\}$ denotes the time-ordered product. It's not difficult to construct scalar interaction density $\mathscr{H}(x)$ out of creation and annihilation operators (7) and (8). The problem arises from the time-ordering of the operator product. A direct way is to require $\mathscr{H}(x)$ all commute at spacelike separations

$$[\mathscr{H}(x), \mathscr{H}(x')] = 0 \;\; \text{for} \;\; (x - x')^2 \geq 0. \tag{11}$$

However this condition is to strong for infinite statistics fields. In our previous work [18] we proved that the relation (9) leads to a weaker locality condition

$$\int d^3y \; \boldsymbol{x} \; [\mathscr{H}(\boldsymbol{x}, 0), \mathscr{H}(\boldsymbol{y}, 0)] = 0. \tag{12}$$

Since the weakest sufficient condition for Lorentz invariance of the S-matrix is given in [19] that $0 = \int d^3x \int d^3y \; \boldsymbol{x} \; [\mathscr{H}(\boldsymbol{x}, 0), \mathscr{H}(\boldsymbol{y}, 0)]$, we conclude that the interaction field theory based on infinite statistics is Lorentz invariant. It's worth noticing that this invariance is only valid for whole structure of S-matrix, and the perturbative terms in Dyson series (10) (from second order to higher) do not hold such invariance. This is not strange to us, because infinite statistics theory is non-local, and we can not limit the number of interaction points, so each order of S-matrix can not solely describe a real physical process. However, we can analyze the perturbative graphs as an analogy tool, as we will see in Sec.V.

Another problem for relativistic theory of infinite statistics fields comes from the condition that the energy of widely spacelike separated subsystems should be additive

$$[\mathscr{H}(x), \psi(x')] \to 0, \;\text{as}\; x - x' \to \infty \;\text{spacelike}. \tag{13}$$

We figured out this problem by introducing a general operator definition $\mathcal{O} \to \mathcal{A}(\mathcal{O})$ (this new definition also satisfies the weak locality condition (12)) in infinite statistics theory

$$\mathcal{A}(\mathcal{O}) \equiv \sum_{m=0}^{\infty} \sum_{n_1,\ldots,n_m,} \sum_{\sigma_1,\cdots,\sigma_m} \int d^3k_1 \cdots d^3k_m \; a_{\boldsymbol{k}_1}^{\dagger(n_1)}(\sigma_1) \cdots a_{\boldsymbol{k}_m}^{\dagger(n_m)}(\sigma_m) \mathcal{O} \; a_{\boldsymbol{k}_m}^{(n_m)}(\sigma_m) \cdots a_{\boldsymbol{k}_1}^{(n_1)}(\sigma_1) \tag{14}$$

and requiring that $\mathscr{H}$ in interaction density $\mathcal{A}(\mathscr{H}(x))$ should have at least one annihilation infinite statistics field and one creation infinite statistics field. Then we can get

$$[\mathcal{A}(\mathscr{H}(x)), \psi(x')] \to 0, \;\text{as}\; x - x' \to \infty \;\text{spacelike}. \tag{15}$$

We also showed this condition imposes conservation of statistics rules, i.e. there must be infinite statistics particles both in the initial and final states when an interaction involves infinite statistics particles.

III. CLUSTER DECOMPOSITION

Cluster decomposition is one of the fundamental principles of physics, which indicates that widely spacelike separated experiments have unrelated results. This is the foundation that we can make predictions about experiments. Greenberg [2, 3] has showed that an arbitrary vacuum matrix element of a product of infinite statistics fields is a sum of products of two-point functions, which means cluster decomposition principle holds for free propagating fields.

In general scattering theory, this principle states that for distant processes $\alpha_1 \to \beta_1$, $\alpha_2 \to \beta_2$, $\cdots$, $\alpha_n \to \beta_n$, the overall S-matrix element can be factorized

$$S_{\beta_1\beta_2\cdots\beta_n,\alpha_1\alpha_2\cdots\alpha_n} \to S_{\beta_1\alpha_1}S_{\beta_2\alpha_2}\cdots S_{\beta_n\alpha_n}, \tag{16}$$

or more generally

$$S_{\beta\alpha} = \sum_{\text{PART}} (\pm) S^C_{\beta_1\alpha_1} S^C_{\beta_2\alpha_2}\cdots . \tag{17}$$

The sum is over all clusters α_1, α_2, $\cdots$ (likewise β_1, β_2, $\cdots$)in the state α (likewise β). The term $S^C_{\beta\alpha}$ with superscript C is the connected part of the S-matrix. In conventional theory the minus sign denotes permutations of odd fermions, however as we will see in the next section, due to CPT symmetry, such sign also involves in infinite statistics spinor interactions.

In order to prove clustering, we can take two steps. First, factorize the particles in the state α, β into different products. Then prove that every term including spatially distant processes vanishes. It's not difficult to realize these in infinite statistics field theory. As we showed in [18], due to the non-local operator form $\mathcal{A}(\mathscr{H})$, the fields in $\mathscr{H}$ can always be transported by the creation or annihilation operators a, $a^\dagger$ in $\int d^3p_i \cdots a^\dagger_{\boldsymbol{p}_i} \cdots \mathscr{H}(x) \cdots a_{\boldsymbol{p}_i} \cdots$ (the creation fields move to the left, and the annihilation fields move to the right). On the other hand, the conservation of statistics rules require that each $\mathscr{H}$ has at least one creation and annihilation fields. So one $\mathscr{H}(x)$ in interaction Hamiltonian density $\mathcal{A}(\mathscr{H}(x))$ at coordinate x can always interact with another $\mathscr{H}(x')$ at coordinate x'. This is our first step. We have also showed that in infinite statistics $\psi^{+(n)}(x)\psi^{-(m)}(x') \sim \delta(nm)\Delta_+(x-x') \to 0$ as $x - x' \to \infty$ spacelike. Since the connection of two $\mathscr{H}$s comes from the contraction between creation and annihilation fields, we can see that the correlations among very distant experiments vanish. So the cluster decomposition principle is also valid in our new theory.

IV. CPT SYMMETRY

As we have seen in Sec. II, the construction of free infinite statistics fields is in the same way as we do in conventional field theory. The only difference is the basic fields here are just creation and annihilation fields. So the CPT transformations of infinite statistics fields are similar to boson (fermion) fields. For a scalar, vector, or spinor field the transformation rule is [19]

$$[\text{CPT}]\phi^{\pm}(\text{x})[\text{CPT}]^{-1} = \phi^{\pm\text{c}}(-\text{x}), \tag{18}$$
$$[\text{CPT}]\phi^{\pm\mu}(\text{x})[\text{CPT}]^{-1} = -\phi^{\pm\mu\text{c}}(-\text{x}), \tag{19}$$
$$[\text{CPT}]\psi^{\pm}(\text{x})[\text{CPT}]^{-1} = -\gamma_5\psi^{\mp\text{c}*}(-\text{x}), \tag{20}$$

in which $\gamma_5 \equiv -i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \gamma^0 = -i\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \boldsymbol{\gamma} = -i\begin{bmatrix}0 & \boldsymbol{\sigma}\\ -\boldsymbol{\sigma} & 0\end{bmatrix}$, the components of $\boldsymbol{\sigma}$ are the usual Pauli matrices. Since a scalar interaction density $\mathcal{A}(\mathscr{H}(x))$ must be formed from tensors with an even total number of spacetime indices, the minus sign in Eq. (19) has no effect on CPT symmetry of $\mathcal{A}(\mathscr{H}(x))$. For scalar and vector fields if every term in $\mathscr{H}$ has a corresponding antiparticle part (e.g. if there is one term $\phi^-\phi^{+c}\phi^+$, then there must also exists another term $\phi^{-c}\phi^+\phi^{+c}$), we have

$$[\text{CPT}]\mathcal{A}(\mathscr{H}(\text{x}))[\text{CPT}]^{-1} = \mathcal{A}(\mathscr{H}(-\text{x})). \tag{21}$$

From Eq. (10) we can see that CPT commutes with the S-matrix, i.e. the CPT theorem holds for scalar and vector fields if we take a natural limit (the particle and antiparticle terms appear symmetrically) on $\mathscr{H}(x)$ in interaction density $\mathcal{A}(\mathscr{H}(x))$.

The case is a bit more complicated for spinor fields. In order to construct scalar interaction densities out of spinor fields, we have to use the bilinear combination $\bar{\psi}_1(x)M\psi_2(x)$, in which $\bar{\psi}^{\pm} \equiv \psi^{\pm\dagger}\beta, \beta \equiv i\gamma^0$, the subscripts $1, 2$ include indices such as annihilation/creation index $\pm$ and particle/antiparticle index c. When the matrix M takes $\mathbf{1}$, γ^μ, $\mathscr{J}^{\mu\nu} \equiv -\frac{1}{4}[\gamma^\mu, \gamma^\nu]$, $\gamma_5\gamma^\mu$, or γ_5, such bilinear transfers as a scalar, vector, tensor, axial vector, and pseudoscalar respectively. By applying Eq. (21) we can get

$$\begin{aligned}
&[\text{CPT}]\,[\bar{\psi}_1(x)M\psi_2(x)][\text{CPT}]^{-1}\\
=&\psi_1^{(-)(c)T}(-x)\gamma_5\beta M^*\gamma_5\psi_2^{(-)(c)*}(-x)\\
=&-\psi_1^{(-)(c)T}(-x)\beta(\gamma_5M^*\gamma_5)\psi_2^{(-)(c)*}(-x),
\end{aligned} \tag{22}$$

in which the superscript $(-)$ denotes the interchange between creation field and annihilation field, the superscript (c) denotes the interchange between particle and antiparticle. The complex conjugate of M in the right hand side of Eq. (22) comes from the antiunitary of CPT. It's worth noticing that $\gamma_5M\gamma_5 = (-1)^nM$, where n is the tensor rank of $\bar{\psi}_1(x)M\psi_2(x)$, since the interaction density should be scalar, the sign $(-1)^n$ does not affect CPT symmetry. Especially, when M takes unit matrix $\mathbf{1}$ or γ_5, we have $M^* = M$, then by using Eq. (22) and considering the hermiticity of interaction density, the general

form of $\mathscr{H}$ to keep CPT invariance is

$$\begin{aligned} & C_1(\psi^{+\dagger}\beta M\psi^{+} - \psi^{-cT}\beta M\psi^{-c*}) \\ & + C_2(\psi^{+\dagger}\beta M\psi^{-c} - \psi^{-cT}\beta M\psi^{+*}) \\ & + C_3(\psi^{-c\dagger}\beta M\psi^{+} - \psi^{+T}\beta M\psi^{-c*}) \\ & + C_4(\psi^{+\dagger}\beta M\psi^{+*} - \psi^{-cT}\beta M\psi^{-c}) \\ & + C_5(\psi^{+T}\beta M\psi^{+} - \psi^{-c*T}\beta M\psi^{-c*}), \end{aligned} \tag{23}$$

in which $C_1, \cdots, C_5$ are coefficients, the minus sign appears in such interactions, just like conventional fermion field theories.

When M takes γ^μ, $\mathscr{J}^{\mu\nu}$ or $\gamma_5\gamma^\mu$, we have $M^* \neq M$ and $\bar{\psi}_1(x)M^*\psi_2(x)$ does not satisfy the Lorentz transformation rules. So the Lorentz invariance indicates CPT violation in such kinds of interactions. This is different with conventional Dirac fields, in which the anticommutation of fermionic operators gives

$$\psi_1^T(-x)\gamma_5\beta M^*\gamma_5\psi_2^*(-x) = [\bar{\psi}_1(-x)\gamma_5 M\gamma_5\psi_2(-x)]^\dagger, \tag{24}$$

then the hermiticity ensures CPT invariance. However Eq. (24) does not hold for infinite statistics fields without commutation relation. Noting that term $\bar{\psi}_1(x)\gamma^\mu\psi_2(x)$ is necessary in vector-spinor couplings, we conclude the "electromagnetic" interactions for infinite statistics field theory have violated CPT symmetry.

V. RENORMALIZATION

In quantum field theory, the normalizability is usually related to the superficial degree of divergence D, which is the actual degree of divergence of the integration over the region of momentum space in which the momenta of all internal lines go to infinity together. By considering the structure of Feynman diagrams (or dimensional analysis) [19], we can get

$$D = 4 - \sum_f E_f(s_f+1) - \sum_i N_i\Delta_i, \tag{25}$$

where E_f denotes the number of external lines of field type f, s_f denotes the spin of field type f, N_i denotes the number of vertices of interaction type i. Δ_i is a parameter characterizing interactions of type i (i.e. the dimensionality of coupling coefficient), $\Delta_i \equiv 4 - d_i - \sum_f n_{if}(s_f+1)$, in which d_i denotes the number of derivatives in each interaction of type i, n_{if} denotes the number of fields of type f in interactions of type i.

Since the term $4 - \sum_f E_f(s_f+1)$ is fixed for a specific physical process, the divergence is mainly depended on Δ_i. If $\Delta_i \geq 0$, then D has a upper bound $D \leq 4 - \sum_f E_f(s_f+1)$. In this case only a finite number of divergences can exists, we can removed such divergences by a renormalization of fields, so we call such theories renormalizable. If $\Delta_i < 0$, the superficial degree of divergence will become larger when more vertices involves, we need an infinite number of couplings to absorb such divergence. Such theories are called non-renormalizable.

As we have shown in [18], the external line term in infinite statistics theory has the same form as in conventional theory, while the internal line term has a non-covariant form [20]

$$\begin{aligned} \Delta_F(q) &\equiv (2\pi)^{-4}\frac{-P_{lm}^{(L)}(q)}{2\sqrt{\boldsymbol{q}^2+m^2}(q^0-\sqrt{\boldsymbol{q}^2+m^2}+i\epsilon)} \\ &= (2\pi)^{-4}\left(\frac{1}{2}+\frac{q^0}{2\sqrt{\boldsymbol{q}^2+m^2}}\right)\frac{P_{lm}^{(L)}(q)}{q^2+m^2-i\epsilon}. \end{aligned} \tag{26}$$

Noting that the propagator has the same dimensionality as conventional theory, especially it has the same behavior near its pole as for a boson/fermion field, so we can use the traditional dimensional analysis method to calculate D and Δ_i for infinite statistics field theory.

We can deal with the finite number of divergences by non-perturbative methods (see Appendix for details). For example, consider a vector particle self-energy process, with two vector external lines and without any spinor external lines. It has $D=2$, the divergent part is a second-order polynomial in q. The complete propagator is given by a sum of one-particle-irreducible (OPI) subgraphs,

$$\begin{aligned} \Delta'_{\mu\nu}(q) &= \Delta_{\mu\nu}(q) + \Delta_{\mu\rho}(q)\Pi^{\rho\sigma}(q)\Delta_{\sigma\nu}(q) \\ &\quad + \Delta_{\mu\rho}(q)\Pi^{\rho\sigma}(q)\Delta_{\sigma\alpha}(q)\Pi^{\alpha\beta}(q)\Delta_{\beta\nu}(q) + \cdots \\ &= [\Delta(q)^{-1} - \Pi(q)]^{-1}_{\mu\nu}, \end{aligned} \tag{27}$$

in which $\Delta_{\mu\nu}(q) = (\frac{1}{2} + \frac{q^0}{2\sqrt{\boldsymbol{q}^2}})\frac{(P_{lm}^{(L)}(q))_{\mu\nu}}{q^2-i\epsilon}$ is the bare vector field propagator, $\Pi^{\mu\nu}(q)$ is the OPI contribution. As we have proved, $\Delta'_{\mu\nu}(q)$ is Lorentz covariant, while $\Delta_{\mu\nu}(q)$ and $\Pi^{\mu\nu}(q)$ is non-covariant. However, we can do a transformation

$$\Delta^*_{\mu\nu}(q) \equiv \frac{(P_{lm}^{(L)}(q))_{\mu\nu}}{q^2-i\epsilon}, \tag{28}$$

$$\begin{aligned} \Pi^{*\mu\nu}(q) &\equiv \Pi^{\mu\nu}(q) + \Delta^*_{\mu\nu}(q)^{-1} - \Delta_{\mu\nu}(q)^{-1} \\ &= \Pi^{\mu\nu}(q) - (\sqrt{\boldsymbol{q}^2}-q^0)^2[(P_{lm}^{(L)}(q))_{\mu\nu}]^{-1}, \end{aligned} \tag{29}$$

then

$$\Delta'_{\mu\nu}(q) = [\Delta(q)^{-1} - \Pi(q)]^{-1}_{\mu\nu} = [\Delta^*(q)^{-1} - \Pi^*(q)]^{-1}_{\mu\nu}. \tag{30}$$

Now both $\Delta^*_{\mu\nu}(q)$ and $\Pi^{*\mu\nu}(q)$ are Lorentz covariant, and we can use this property and $q_\mu\Pi^{*\mu\nu}=0$ to write the form of $\Pi(q)$

$$\begin{aligned} \Pi^{\mu\nu}(q) =& (\eta^{\mu\nu}q^2 - q^\mu q^\nu)(\pi(q^2)) \\ & - (\sqrt{\boldsymbol{q}^2}-q^0)^2[(P_{lm}^{(L)}(q))^{-1}]^{\mu\nu}, \end{aligned} \tag{31}$$

The non-covariant term is finite and become zero on the mass-shell. Since the introducing of OPI should not change the structure of the pole at $q^2 = 0$, so $\pi(0) = 0$. Now we can renormalize the vector field $A^\mu \to Z^{-1/2}A^\mu$, then $\pi(q^2) = 1 - Z + \pi_{\rm LOOP}(q^2)$ and we can get $Z = 1 + \pi_{\rm LOOP}(0)$.

We see that the divergence in such vector-spinor interaction graph is absorbed by coupling constant Z. One can easily check that the self-energy process in scalar electrodynamics (in which CPT theorem holds) can also be renormalized in a similar way.

VI. CONCLUSIONS

Quantum field theory based on infinite statistics is a valid relativistic theory. In this paper, we analyze the self-consistency of such theory. Although this is a non-local theory, it obeys cluster decomposition principle. If we take a limit that the particle and antiparticle terms appear symmetrically in the interaction density, the CPT theorem also holds when we realize infinite statistics theory by scalar fields, vector fields or scalar (pseudoscalar) spinor-pairs $\bar\psi_1\psi_2$ ($\bar\psi_1\gamma_5\psi_2$). However, Lorentz invariance indicates CPT violation in interactions involving vector (axial vector, tensor) spinor-pairs $\bar\psi_1\gamma^\mu\psi_2$ ($\bar\psi_1\gamma_5\gamma^\mu\psi_2$, $\bar\psi_1\mathscr{J}^{\mu\nu}\psi_2$), which means that the "electromagnetic" interactions have violated CPT symmetry. We also showed that this theory is renormalizable through dimension analysis and non-perturbative methods.

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Appendix: Non-Perturbative Methods

In this Appendix, we give a non-perturbative method which will be useful in deriving results valid to whole S-matrix (beyond perturbation theory). The analysis is analog to Chap. 10 in [19], however one should keep in mind the difference between infinite statistics theory and conventional field theory.

Consider a momentum-space amplitude

$$G(q_1\cdots q_n) \equiv \int d^4x_1\cdots d^4x_n e^{-iq_1\cdot x_1}\cdots e^{-iq_n\cdot x_n} \langle T\{A_1(x_1)\cdots A_n(x_n)\}\rangle_0, \tag{A.1}$$

where As are Heisenberg-picture operators and $\langle\cdots\rangle_0$ denotes the expectation value in the true vacuum. By using Fourier representation of the step function $\theta(\tau) = -\frac{1}{2\pi i}\int_\infty^\infty \frac{d\omega \exp -i\omega\tau}{\omega+i\epsilon}$ and spacetime translational invariance, we can show that near the pole (see Chap. 10 in [19] for details)

$$G \to \frac{-2i\sqrt{\boldsymbol{q}^2+m^2}}{q^2+m^2-i\epsilon}(2\pi)^7\delta^4(q_1+\cdots+q_n) \sum_\sigma M_{0|\boldsymbol{q},\sigma}(q_2\cdots q_r)M_{\boldsymbol{q},\sigma|0}(q_{r+2}\cdots q_n) + \mathrm{OT}, \tag{A.2}$$

where

$$q \equiv q_1+\cdots+q_r = -q_{r+1}-\cdots-q_n,\ 1\le r\le n-1, \tag{A.3}$$

$$\begin{aligned}
&\int d^4x_1\cdots d^4x_r e^{-iq_1\cdot x_1}\cdots e^{-iq_r\cdot x_r}\\
&\times(\Psi_0, T\{A_1(x_1)\cdots A_r(x_r)\}\Psi_{\boldsymbol{p},\sigma})\\
=&(2\pi)^4\delta^4(q_1+\cdots+q_r-p)M_{0|\boldsymbol{p},\sigma}(q_2\cdots q_r),\\
&\int d^4x_{r+1}\cdots d^4x_n e^{-iq_{r+1}\cdot x_{r+1}}\cdots e^{-iq_n\cdot x_n}\\
&\times(\Psi_{\boldsymbol{p},\sigma}, T\{A_{r+1}(x_{r+1})\cdots A_n(x_n)\}\Psi_0)\\
=&(2\pi)^4\delta^4(q_{r+1}+\cdots+q_n-p)M_{\boldsymbol{p},\sigma|0}(q_{r+2}\cdots q_n).
\end{aligned} \tag{A.4}$$

'OT' denotes other terms that exhibit different poles. This calculation is irrelative to the kind of particle statistics, and the pole structure of G is quite like the pole in a Feynman diagram with a single internal line. In conventional theory, $M_{0|\boldsymbol{p},\sigma}(q_2\cdots q_r)$ is explained as r external lines (with the factor $(2\pi)^{-2/3}u_l$ stripped away). Since $A(x)$ need not be some elementary particle field, it may be also a bound state, so the pole arises not from single Feynman diagrams and G can be seen as a contribution of infinite sums of diagram. While in infinite statistics theory, the time-ordered product in M is usually not Lorentz covariant, so we can not directly explain it as the external line term. However we can further reduce Eq. (A.2) to

$$\begin{aligned}
G \to\ & \delta^4(q_1+\cdots+q_n)\\
&\times\sum_{\sigma_1,\cdots,\sigma_{n-1}}(\Psi_0, A_1(0)\Psi_{\boldsymbol{q}_1,\sigma_1})(\Psi_{\boldsymbol{q}_1,\sigma_1}, A_2(0)\Psi_{\boldsymbol{q}_1+\boldsymbol{q}_2,\sigma_1})\cdots\\
&\times(\Psi_{\boldsymbol{q}_1+\cdots+\boldsymbol{q}_{n-1},\sigma_{n-1}}, A_n(0)\Psi_0)\\
&\times\frac{-2i\sqrt{\boldsymbol{q_1}^2+m_1^2}}{q_1^2+m_1^2-i\epsilon}\cdot\frac{-2i\sqrt{(\boldsymbol{q}_1+\boldsymbol{q}_2)^2+m_2^2}}{(q_1+q_2)^2+m_2^2-i\epsilon}\cdots\\
&\times\frac{-2i\sqrt{(\boldsymbol{q}_1+\cdots+\boldsymbol{q}_{n-1})^2+m_{n-1}^2}}{(q_1+\cdots q_{n-1})^2+m_{n-1}^2-i\epsilon}+\mathrm{OT},
\end{aligned} \tag{A.5}$$

in which every matrix element contains only one operator and should be Lorentz covariant. So Eq. (A.5) can be seen as a tree level non-perturbative diagram with n external lines.

First we see the $n=2$ case. Eq. (A.5) becomes

$$\begin{aligned}&\int d^4x_1 d^4x_2 e^{-iq_1\cdot x_1}e^{-iq_2\cdot x_2}(\Psi_0, T\{\mathcal{O}_l(x_1)\mathcal{O}^\dagger_{l'}(x_2)\}\Psi_0)\\ &\xrightarrow{q_1^0=\sqrt{\boldsymbol{q}_1^2+m^2}}(2\pi)^7\frac{-2i\sqrt{\boldsymbol{q}_1^2+m^2}}{q_1^2+m^2-i\epsilon}\\ &\times\sum_\sigma(\Psi_0,\mathcal{O}_l(0)\Psi_{\boldsymbol{q}_1,\sigma})(\Psi_{\boldsymbol{q}_1,\sigma},\mathcal{O}^\dagger_{l'}(0)\Psi_0)\delta^4(q_1+q_2)\\ &=\frac{-2i|N|^2\sqrt{\boldsymbol{q}_1^2+m^2}}{q_1^2+m^2-i\epsilon}\sum_\sigma u_l(\boldsymbol{q}_1,\sigma)u^*_{l'}(\boldsymbol{q}_1,\sigma)(2\pi)^4\delta^4(q_1+q_2),\end{aligned} \tag{A.6}$$

in which $\mathcal{O}(x)$ is a Heisenberg-picture operator with the same Lorentz transformation properties as some sort of free field ψ_l, so we can write matrix element

$$(\Psi_0,\mathcal{O}_l(0)\Psi_{\boldsymbol{q},\sigma})=(2\pi)^{-3/2}Nu_l(\boldsymbol{q},\sigma). \tag{A.7}$$

Eq. (A.6) is just the usual behavior of a propagator near its pole except for the factor $|N|^2$, and we can redefine a normalized field to absorb this factor

$$(\Psi_0,\psi_l(0)\Psi_{\boldsymbol{q},\sigma})=(2\pi)^{-3/2}u_l(\boldsymbol{q},\sigma). \tag{A.8}$$

This normalized field is called a renormalized field. On the other hand the mass m in the pole also need not to correspond to a elementary field, so we call m the renormalized mass, which is defined by the position of the pole.

Here we take a theory of scalar field for example, the free field is Ψ_B, with mass m_B. By introducing a renormalized field $\Phi\equiv Z^{-1/2}\Phi_B$ and mass $m^2\equiv m_B^2+\delta m^2$, we can rewrite the Lagrangian density as

$$\begin{aligned}&\mathscr{L}=\mathscr{L}_0+\mathscr{L}_1,\\ &\mathscr{L}_0=-\partial_\mu\Phi^-\partial^\mu\Phi^+-m^2\Phi^-\Phi^+\\ &\mathscr{L}_1=-(Z-1)[\partial_\mu\Phi^-\partial^\mu\Phi^++m^2\Phi^-\Phi^+]\\ &\qquad+Z\delta m^2\Phi^-\Phi^+-V(\Phi).\end{aligned} \tag{A.9}$$

Now we can use OPI to calculate the full propagator

$$\begin{aligned}\Delta'(q)=&[-2\sqrt{\boldsymbol{q}^2+m^2}(q^0-\sqrt{\boldsymbol{q}^2+m^2}+i\epsilon)]^{-1}\\ &+[-2\sqrt{\boldsymbol{q}^2+m^2}(q^0-\sqrt{\boldsymbol{q}^2+m^2}+i\epsilon)]^{-1}\Pi(q)\\ &\times[-2\sqrt{\boldsymbol{q}^2+m^2}(q^0-\sqrt{\boldsymbol{q}^2+m^2}+i\epsilon)]^{-1}+\cdots\\ =&[-2\sqrt{\boldsymbol{q}^2+m^2}(q^0-\sqrt{\boldsymbol{q}^2+m^2}+i\epsilon)-\Pi(q)]^{-1}.\end{aligned} \tag{A.10}$$

In calculating OPI graph Π, we encounter a vertices contribution arising from $\mathscr{L}_1$, plus a loop term Π_{LOOP}

$$\Pi(q)=-(Z-1)(q^2+m^2)+Z\delta m^2+\Pi_{\text{LOOP}}(q). \tag{A.11}$$

We can rewrite Eq. (A10) by Lorentz covariant terms $\Delta^*(q)=(q^2+m^2-i\epsilon)$ and $\Pi^*(q^2)$

$$\Delta'(q)=[q^2+m^2-\Pi^*(q^2)-i\epsilon]^{-1}, \tag{A.12}$$

where

$$\begin{aligned}\Pi^*(q^2)=&-(Z-1)(q^2+m^2)+Z\delta m^2+\Pi_{\text{LOOP}}(q)\\ &-(\sqrt{\boldsymbol{q}^2+m^2}-q^0)^2.\end{aligned} \tag{A.13}$$

The non-covariant term $-(\sqrt{\boldsymbol{q}^2+m^2}-q^0)^2$ do not contribute near the pole. In order to make the pole of the propagator be at $q^2=-m^2$ and has a unit residue, we must have

$$\begin{aligned}&\Pi^*(-m^2)=0,\\ &[\frac{d}{dq^2}\Pi^*(q^2)]_{q^2=-m^2}=0,\end{aligned} \tag{A.14}$$

which leads to

$$\begin{aligned}&Z\delta m^2=-[\Pi_{\text{LOOP}}(q)]_{q^0=\sqrt{\boldsymbol{q}^2+m^2}},\\ &Z=1+[\frac{d}{dq^0}\Pi_{\text{LOOP}}(q)]_{q^0=\sqrt{\boldsymbol{q}^2+m^2}}.\end{aligned} \tag{A.15}$$

Similar analyses can also be applied to particles of arbitrary spin.

Then we see the $n=3$ case, one typical example is vertex correction. We can define the vertex function Γ^μ of the charged particle by

$$\begin{aligned}&\int d^4xd^4yd^4ze^{-ip\cdot x}e^{-ik\cdot y}e^{+il\cdot z}(\Psi_0,T\{J^\mu(x)\Psi_n^+(y)\bar{\Psi}_m^+(z)\}\Psi_0)\\ &\quad\to -i(2\pi)^4qS'_{nn'}(k)\Gamma^\mu_{n'm'}(k,l)S'_{m'm}(l)\delta^4(p+k-l),\end{aligned} \tag{A.16}$$

in which q is charge and S' is the complete spinor propagator

$$\begin{aligned}&-i(2\pi)^4S'_{nm}(k)\delta^4(k-l)\\ &\leftarrow\int d^4yd^4z(\Psi_0,T\{\Psi_n^+(y)\bar{\Psi}_m^+(z)\}\Psi_0)e^{-ik\cdot y}e^{+il\cdot z}.\end{aligned} \tag{A.17}$$

We can see that Γ^μ is the sum of vertex graphs with one incoming spinor line, one outgoing spinor line and one vector line (with the external line coefficients stripped away), in the limit of no interactions this becomes γ^μ.

In conventional field theory, we can give a relation between Γ^μ and S' through Ward identity. Here we will show that a similar relation also exists in infinite statistics theory. First we have

$$\begin{aligned}\frac{\partial}{\partial x^\mu}T\{J^\mu(x)\Psi_n^+(y)\bar{\Psi}_m^+(z)\}=&T\{\partial_\mu J^\mu(x)\Psi_n^+(y)\bar{\Psi}_m^+(z)\}\\ &+\delta(x^0-y^0)T\{[J^0(x),\Psi_n^+(y)]\bar{\Psi}_m^+(z)\}\\ &+\delta(x^0-z^0)T\{\Psi_n^+(y)[J^0(x),\bar{\Psi}_m^+(z)]\},\end{aligned} \tag{A.18}$$

where the delta functions arise from time-derivatives of step functions. The first term of the right hand side vanishes by the conservation condition $\partial_\mu J^\mu=0$. From Eqs. (A.16) and (A.17) we can see that the real part in J^μ that gives contributions is $-iq\bar{\Psi}^+\Gamma^\mu\Psi^+$, so

$$\begin{aligned}\left[J^0(\boldsymbol{x},t),\Psi_n^+(\boldsymbol{y},t)\right]=&-q\Psi^{+\dagger}(\boldsymbol{x},t)\Psi^+(\boldsymbol{x},t)\Psi_n^+(\boldsymbol{y},t)\\ &+q\Psi_n^+(\boldsymbol{y},t)\Psi^{+\dagger}(\boldsymbol{x},t)\Psi^+(\boldsymbol{x},t),\end{aligned} \tag{A.19}$$

The first term will be annihilated by the vacuum state when substituted into Eqs. (A.18) and (A.16), while the second term becomes $-q\Psi_n(\boldsymbol{y},t)\delta^3(\boldsymbol{x}-\boldsymbol{y})$ on the mass shell, so we can get

$$\begin{aligned}\left[J^0(\boldsymbol{x},t),\Psi_n^+(\boldsymbol{y},t)\right] &\to -q\Psi_n^+(\boldsymbol{y},t)\delta^3(\boldsymbol{x}-\boldsymbol{y}),\\ \left[J^0(\boldsymbol{x},t),\bar{\Psi}_n^+(\boldsymbol{y},t)\right] &\to q\bar{\Psi}_n^+(\boldsymbol{y},t)\delta^3(\boldsymbol{x}-\boldsymbol{y}).\end{aligned}\tag{A.20}$$

Substituting this into Eq. (A.18) gives

$$\begin{aligned}\frac{\partial}{\partial x^\mu}T\{J^\mu(x)\Psi_n^+(y)\bar{\Psi}_m^+(z)\} &\to -q\delta^4(x-y)T\{\Psi_n^+(y)\bar{\Psi}_m^+(z)\}\\ &+q\delta^4(x-z)T\{\Psi_n^+(y)\bar{\Psi}_m^+(z)\}.\end{aligned}\tag{A.21}$$

From Fourier transform (A.16) we have Ward identity

$$(l-k)_\mu\Gamma^\mu(k,l)=iS'^{-1}(k)-iS'^{-1}(l).\tag{A.22}$$

In the $l\to k$ limit, this gives

$$\Gamma^\mu(k,k)=-i\frac{\partial}{\partial k_\mu}S'^{-1}(k).\tag{A.23}$$

As we discussed above, S' can be written as a Lorentz covariant form

$$S'^{-1}=i\partial\!\!\!/ k\!\!\!/+m-\Sigma^*(k\!\!\!/),\tag{A.24}$$

in which $k\!\!\!/=k^\mu\gamma_\mu$, then Eq. (A.23) becomes

$$\Gamma^\mu(k,k)=\gamma^\mu+i\frac{\partial}{\partial k_\mu}\Sigma^*(k\!\!\!/).\tag{A.25}$$

The condition that the pole of S' has a unit residue requires $\frac{\Sigma^*(k\!\!\!/)}{\partial k\!\!\!/}|_{k\!\!\!/=im}=0$, so on the mass shell we have

$$\bar{u}'_k\Gamma^\mu(k,k)u_k=\bar{u}'_k\gamma^\mu u_k,\tag{A.26}$$

in which $[i\gamma_\mu k^\mu+m]u_k=[i\gamma_\mu k^\mu+m]u'_k=0$.

For the vector particle self-energy graph, there is a bit different. In conventional field theory, we can introduce

$$M^{\mu\nu}(q)=\int d^4xd^4ye^{-iq\cdot x}e^{-iq'\cdot y}(\Psi_0,T\{J^\mu(x)J^\nu(y)\}\Psi_0)\tag{A.27}$$

to construct the complete propagator Δ'

$$\begin{aligned}\Delta'_{\mu\nu}(q)&=\Delta_{\mu\nu}+\Delta_{\mu\rho}(q)M^{\rho\sigma}(q)\Delta_{\sigma\nu}\\ &=\Delta_{\mu\nu}+\Delta_{\mu\rho}(q)\Pi^{\rho\sigma}(q)\Delta'_{\sigma\nu}.\end{aligned}\tag{A.28}$$

From current conservation and local commutativity, we can get $q^\mu M_{\mu\nu}(q)=0$, then from the specific form of $\Delta_{\mu\nu}$, we have $q^\mu\Delta'_{\mu\nu}(q)=q^\mu\Delta_{\mu\nu}(q)$, which leads to $q_\rho\Pi^{\rho\sigma}(q)=0$. While in infinite statistics theory, we can not solve the problem in the same way because of the lake of locality. However, since the Lorentz covariant part in each order of Feynman graph has the same form as that in conventional theory and vanishes when contracting with q, we guess that for infinite statistics OPI, the covariant part $\Pi^*(q)$ satisfies $q_\rho\Pi^{*\rho\sigma}(q)=0$, then we

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[20] We apologize that we made a mistake in Eq. (28) in [18]. We exchanged x,y in Eq. (27) in [18] twice and got a wrong form of propagator. The right form should be Eq. (26) here. This error does not affect the main conclusion of the article.